International Journal of Theoretical Physics, Vol. 1, No. 1 (1968), pp. 37–50.

Fundamental Limitations on the Detectability of Electromagnetic Signals

CARL W. HELSTROM

Department of Applied Electrophysics, University of California, San Diego, La Jolla, California, and Institute for Radiation Physics and Aerodynamics†

Abstract

The detection of signals in an ideal receiver is considered as a decision between quantum-mechanical density operators describing the field in the receiver, respectively when thermal radiation alone is present and when besides this background there is a signal field present. The detectability of the signal is assessed by the minimum probability of error attainable in such a decision. Coherent signals of known and unknown phase are treated. The theory is shown to reduce to conventional detection theory in the classical limit.

1. *Detection as Measurement and Decision*

The limitations imposed by the laws of nature on the detectability of electromagnetic signals are of both philosophical and practical interest. To determine them one defines an ideal receiver of such a signal and carries out a thought experiment.

The ideal receiver is taken to be a lossless cavity containing thermal radiation of absolute temperature T. The detectability of a signal can be assessed by imagining a large number of random trials, in a fraction $(1 - \zeta)$ of which the receiver contains also an electromagnetic field due to the signal. In the remaining fraction ζ of the trials, the signal field is absent. In each trial the observer measures the total field in the cavity as best he can, and on the basis of his results he decides between one of two hypotheses, (H_0) 'the cavity contains only thermal radiation' and (H_1) 'the cavity contains also a field due to the signal'. Because of the random nature of the thermal radiation, some of these decisions will be in error, and the detectability of this signal is specified by the minimum possible average probability of error, which will be a function of the energy in the signal, the temperature T of the cavity,

† This research was supported by the Advanced Research Projects Agency (Project DEFENDER) and was monitored by the U.S. Army Research Office, Durham, under Contract DA-31-124-ARO-D-257.

the prior probabilities ζ and $(1-\zeta)$ of H_0 and H_1, and, perhaps, other parameters of the signal and the background radiation.

Classically the total field can be measured completely, and probability density functions can be written for the field variables at an arbitrarily large number of points under both hypotheses. Conventional detection theory, based on the theory of statistical decisions, presupposes this classical description. When the quantum-mechanical character of the field must be taken into account, however, as at optical frequencies, this approach fails. The present paper will show how the theory can be reformulated to embody the quantum-mechanical limitations on measurement.

The most elementary situation of a coherent signal field will be treated first from a classical standpoint. Then we shall describe how the thought experiment can be carried out in the framework of quantum mechanics. The results that have been obtained thus far will be summarized. Finally, detection of a coherent signal of unknown phase, which is more realistic at optical frequencies, will be analyzed.

2. *The Classical Limitations on Detectability*

For simplicity we shall describe the electromagnetic field in the cavity by a scalar function $\varphi(\mathbf{r},t)$ satisfying the wave equation with appropriate boundary conditions at the walls. It will be apparent that a vector field can be similarly treated; the results will be the same, the mathematics only somewhat more cumbersome. We express the field in terms of normal modes $u_n(\mathbf{r})$, which are solutions of the scalar Helmholtz equation,

$$\varphi(\mathbf{r},t) = \sum_n q_{\mathrm{n}}(t)\, u_{\mathrm{n}}(\mathbf{r}) \tag{2.1}$$

The rate of change of the field specifies the conjugate field variable

$$\frac{\partial\varphi}{\partial t} = \pi(\mathbf{r},t) = \sum_n p_{\mathrm{n}}(t)\, u_{\mathrm{n}}(\mathbf{r}) \tag{2.2}$$

The functions $q_n(t)$ and $p_n(t)$ are harmonic functions of the time. In terms of these functions the energy of the field is given by

$$H = \tfrac{1}{2}\int [\pi^2 + (\nabla\varphi)^2]\, d^3\mathbf{r} = \tfrac{1}{2}\sum_n ({p_n}^2 + {\omega_n}^2 {q_n}^2) \tag{2.3}$$

where ω_n is the angular frequency of the nth mode. The integral is taken over the volume of the cavity, and with respect to this integration, the mode functions $u_n(\mathbf{r})$ are orthonormal. The variables q_n and

p_n can be thought of as the coordinate and momentum of a harmonic oscillator of frequency ω_n.†

When the cavity is in thermal equilibrium at absolute temperature T, the mode coordinates q_n and their conjugate momenta p_n become random variables with mean values zero and variances given by the law of equipartition of energy,

$$\tfrac{1}{2}\operatorname{Var} p_n = \tfrac{1}{2}\omega_n{}^2 \operatorname{Var} q_n = kT/2 \tag{2·4}$$

where k is Boltzmann's constant. The joint probability density functions of these variables for an arbitrary finite subset of oscillators is given by the entropy principle in the Gaussian form (Chandrasekhar, 1943)

$$P_0(\{p_n, q_n\}) = M \exp(-H/kT) = M \exp\left[-(2kT)^{-1} \sum_n (p_n{}^2 + \omega_n{}^2 q_n{}^2)\right] \tag{2.5}$$

where the constant M normalizes the density function so that its integral over the infinite ranges of all the variables equals 1.

When a signal is present (hypothesis H_1), the variables p_n and q_n will at any instant of time have mean values P_n and Q_n that are the momenta and coordinates of the signal field itself, and their joint probability density function is

$$P_1(\{p_n, q_n\}) = M \exp\left\{-(2kT)^{-1} \sum_n [(p_n - P_n)^2 + \omega_n{}^2(q_n - Q_n)^2]\right\} \tag{2.6}$$

with the same normalization constant (Chandrasekhar, 1943).

The task of the observer can be regarded as one of deciding which of the probability density functions $P_0(\{p_n, q_n\})$ or $P_1(\{p_n, q_n\})$ provides the better description of the field in the cavity. According to statistical decision theory, the minimum average probability of error in these decisions is attained when the decisions are made on the basis of the likelihood ratio (Helstrom, 1960a; Middleton, 1960)

$$\begin{aligned}\Lambda(\{p_n, q_n\}) &= \frac{P_1(\{p_n, q_n\})}{P_0(\{p_n, q_n\})} \\ &= \exp\left\{(kT)^{-1} \sum_n [(P_n p_n + \omega_n{}^2 Q_n q_n) \right. \\ &\qquad \left. - \tfrac{1}{2}(P_n{}^2 + \omega_n{}^2 Q_n{}^2)]\right\}\end{aligned} \tag{2.7}$$

This likelihood ratio is compared with a decision level $\lambda = \zeta/(1-\zeta)$, where ζ is the prior probability of the hypothesis H_0. If the likelihood

† See, for instance, Wentzel, G. (1949). *Quantum Theory of Fields*, Chapter I. Interscience Publishers, Inc., New York.

ratio exceeds the level λ, the observer decides that a signal is present. Since $\ln \Lambda$ is a Gaussian random variable, it is not hard to show that the minimum probability of error is given by

$$\begin{aligned} p_{e,\min} &= \zeta \Pr\{\Lambda > \lambda | H_0\} + (1-\zeta)\Pr\{\Lambda < \lambda | H_1\} \\ &= \zeta \operatorname{erfc}(\tfrac{1}{2}D + D^{-1}\ln\lambda) + (1-\zeta)\operatorname{erfc}(\tfrac{1}{2}D - D^{-1}\ln\lambda) \end{aligned} \tag{2.8}$$

where

$$\operatorname{erfc} x = (2\pi)^{-1/2} \int_x^\infty \exp(-t^2/2)\,dt \tag{2.9}$$

is the error-function integral and

$$D = (2E_s/kT)^{1/2} \tag{2.10}$$

is the signal-to-noise ratio,

$$E_s = \tfrac{1}{2} \sum_n ({P_n}^2 + {\omega_n}^2 {Q_n}^2) \tag{2.11}$$

being the total energy in the signal field. Thus the detectability of the signal depends primarily on the ratio of twice the energy in the signal to the total average thermal energy per mode, kT.

3. *The Quantum-Mechanical Detection Problem*

The classical description of the field is valid only when the average number of quanta in each oscillator is large under both hypotheses, that is, generally speaking, for angular frequencies ω_n much lower than $kT/\hbar$, where $\hbar = h/2\pi$ is Planck's constant. When the principal signal frequencies are higher than $kT/\hbar$, as in the optical domain, the field must be treated by quantum mechanics. The conjugate variables p_n, q_n of the field oscillators cannot be measured simultaneously, and joint probability density functions of the kind given by equations (2.5) and (2.6) become meaningless. The likelihood ratio cannot be written down, and conventional detection theory is inapplicable. The problem of how to attain a minimum probability of error on the basis of observations of the field in the ideal receiver must be reexamined (Helstrom, 1967a).

Quantum-mechanically the field must be described not in terms of probability density functions, but in terms of density operators ρ_0 and ρ_1 pertaining respectively to the absence and the presence of the signal. The oscillators are now, under each hypothesis, in a statistical mixture of states, and the observer must decide which of the two possible mixtures better fits what measurements he can make of the field. His decisions will sometimes be in error, and it is necessary to determine what are the most effective measurements to make and how best can their outcomes be processed so that the probability of error will be minimum. The minimum attainable probability of error, as before, specifies the detectability of the signal.

At most the observer can measure a set of dynamical variables of the field whose quantum-mechanical operators X_1, X_2, X_3, ... commute. He must then form some function $f(x_1, x_2, x_3, \ldots)$ of the outcomes x_1, x_2, x_3, ... of these measurements, and on the basis of the value of this function he chooses hypothesis H_0 or H_1. Equivalent to this is the measurement simply of the operator $\Pi = f(X_1, X_2, X_3, \ldots)$, and the only values that the outcome of such a measurement needs to take are the numbers 0 and 1. If measurement of the operator Π yields the value 0, hypothesis H_0 is chosen; if 1, H_1 is chosen. The operator Π is therefore a projection operator for the system consisting of the field in the ideal receiver.

The remaining problem is to adopt the most effective projection operator Π, that is, the one minimising the average probability of error. The probability of choosing H_1 when H_0 is true is

$$\Pr\{\Pi \to 1|H_0\} = \mathbf{E}(\Pi|H_0) = \mathrm{Tr}\,(\rho_0 \Pi) \tag{3.1}$$

where $\mathbf{E}$ denotes the expected value, and Tr denotes the trace. The probability of choosing H_0 when H_1 is true is similarly

$$\Pr\{\Pi \to 0|H_1\} = \mathbf{E}(1 - \Pi|H_1) = 1 - \mathrm{Tr}\,(\rho_1 \Pi) \tag{3.2}$$

Thus with ζ again the prior probability of H_0, the average probability of error is

$$\begin{aligned} P_e &= \zeta\,\mathrm{Tr}\,(\rho_0 \Pi) + (1-\zeta)\,[1 - \mathrm{Tr}\,(\rho_1 \Pi)] \\ &= (1-\zeta) - (1-\zeta)\,\mathrm{Tr}\,(\rho_1 - \lambda\rho_0)\,\Pi \qquad (3.3) \\ \lambda &= \zeta/(1-\zeta) \end{aligned}$$

The observer should therefore measure the projection operator Π for which $\mathrm{Tr}\,(\rho_1 - \lambda\rho_0)\,\Pi$ is maximum.

By taking a representation in which $\rho_1 - \lambda\rho_0$ is diagonal, it is not hard to show that the projection operator satisfying this requirement is given by (Helstrom, 1967a)

$$\Pi = \sum_{\substack{k: \\ \eta_k \geq 0}} |\eta_k\rangle\langle\eta_k| \tag{3.4}$$

where $|\eta_k\rangle$ is the eigenket of the operator $\rho_1 - \lambda\rho_0$ corresponding to the eigenvalue η_k,

$$(\rho_1 - \lambda\rho_0)\,|\eta_k\rangle = \eta_k|\eta_k\rangle \tag{3.5}$$

The projection operator Π projects all state vectors onto the subspace spanned by eigenkets $|\eta_k\rangle$ with positive eigenvalues η_k. The minimum probability of error is then given by

$$P_{e,\min} = (1-\zeta)\left(1 - \sum_{\eta_k>0} \eta_k\right) \tag{3.6}$$

One can say that the observer measures the operator $\rho_1 - \lambda\rho_0$ and chooses hypothesis H_1 if the outcome is positive.

Let the temporal behavior of the system be described by the unitary operator $U(t, t_0)$, which is determined by the Schrödinger equation. Then the density operators at time t in terms of those at an earlier time t_0 are

$$\rho_k(t) = U(t, t_0)\, \rho_k(t_0)\, U^+(t, t_0) \quad (k = 0, 1), \tag{3.7}$$
$$U(t, t_0)\, U^+(t, t_0) = 1$$

and the eigenkets $|\eta_k(t)\rangle$ in terms of those at time t_0 are

$$|\eta_k(t)\rangle = U(t, t_0)\, |\eta_k(t_0)\rangle \tag{3.8}$$

The eigenvalues η_k are unchanged, and the detection operator at time t is

$$\Pi(t) = U(t, t_0)\, \Pi(t_0)\, U^+(t, t_0) \tag{3.9}$$

in terms of that at time t_0. Hence

$$\mathrm{Tr}\,[\rho_1(t) - \lambda\rho_0(t)]\,\Pi(t) = \mathrm{Tr}\,[\rho_1(t_0) - \lambda\rho_0(t_0)]\,\Pi(t_0) \tag{3.10}$$

and the minimum attainable probability of error does not depend on when the measurements are made.

When the density operators ρ_0 and ρ_1 commute, they possess a common array of eigenkets $|\eta_k\rangle$, and the eigenvalues of $\rho_1 - \lambda\rho_0$ are

$$\eta_k = P_{1k} - \lambda P_{0k} \tag{3.11}$$

where P_{jk} is the probability that the receiver is in the kth state under hypothesis H_j $(j = 0, 1)$. The observer then measures any observable having the same set of eigenstates $|\eta_k\rangle$, and he chooses hypothesis H_1 if the state after the measurement is an eigenket $|\eta_k\rangle$ with $\eta_k = P_{1k} - \lambda P_{0k} \geqslant 0$, that is, with

$$\frac{P_{1k}}{P_{0k}} \geqslant \lambda \tag{3.12}$$

This is the standard likelihood-ratio test. If the operators ρ_0 and ρ_1 have continuous spectra of eigenvalues, the probabilities P_{jk} become proportional to probability density functions. Since classically all density operators must commute, the theory reduces in the classical limit to the conventional form of detection theory.

If under hypothesis H_0 the system is in a pure state $|\Psi_0\rangle$, and if under hypothesis H_1 it is in a pure state $|\Psi_1\rangle$, the detection operator Π is simply a projection on to a linear combination of $|\Psi_0\rangle$ and $|\Psi_1\rangle$ (Helstrom, 1967c),

$$\Pi = |\eta_1\rangle\langle\eta_1|, \qquad |\eta_1\rangle = x_0|\Psi_0\rangle + x_1|\Psi_1\rangle \tag{3.13}$$

By substituting such a linear combination into equation (3.5), with now

$$\rho_0 = |\Psi_0\rangle\langle\Psi_0|, \qquad \rho_1 = |\Psi_1\rangle\langle\Psi_1| \tag{3.14}$$

and equating coefficients of $|\Psi_0\rangle$ and $|\Psi_1\rangle$, one obtains a pair of simultaneous equations for x_0 and x_1, a solution of which exists only when η_k is a root of a certain determinantal equation. There are two roots, $\eta_0 < 0$ and $\eta_1 > 0$, where

$$\begin{aligned} \eta_{1,\,0} &= \tfrac{1}{2}(1-\lambda) \pm R \\ R &= \{[\tfrac{1}{2}(1-\lambda)]^2 + \lambda q\}^{1/2} \\ q &= 1 - |\langle\Psi_0|\Psi_1\rangle|^2 \end{aligned} \tag{3.15}$$

The minimum probability of error is

$$P_{\mathrm{e,\,min}} = (1-\zeta)(1-\eta_1) = (1-\zeta)[\tfrac{1}{2}(1+\lambda) - R] \tag{3.16}$$

For orthogonal states, $q = 1$ and $P_{\mathrm{e,min}} = 0$, as one would expect. These formulas give the probability of error in deciding between two pure quantum-mechanical states.

4. *Quantum Detection of a Coherent Signal*

Let us suppose that the signal excites only a single mode of oscillation of the field in the receiver. The remaining modes can then be disregarded. If under hypothesis H_0 the mode is excited by thermal radiation, the density operator ρ_0 can be written in the alternative forms (Glauber, 1963a; Louisell, 1964a)

$$\rho_0 = (1-\mathrm{e}^{-w})\exp(-wa^+a) = (\pi N)^{-1}\int \exp(-|\alpha|^2/N)\,|\alpha\rangle\langle\alpha|\,d^2\alpha \tag{4.1}$$

$$w = \hbar\omega/kT, \quad N = (\mathrm{e}^w - 1)^{-1}$$

where ω is the frequency of the mode and N is the average number of thermal photons as given by the Planck law. In equation (4.1), a and a^+ are the annihilation and creation operators for the mode, obeying the usual commutation rule $aa^+ - a^+a = 1$. In terms of them, the oscillator coordinate q and its conjugate momentum p are

$$q = (\hbar/2\omega)^{1/2}(a^+ + a)$$
$$p = i(\hbar\omega/2)^{1/2}(a^+ - a) \tag{4.2}$$

The second form of ρ_0 in equation (4.1) is Glauber's P-representation in terms of the coherent states $|\alpha\rangle$, which are the right-eigenkets of the annihilation operator (Glauber, 1963a)

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{4.3}$$

The quantum-mechanical counterpart of a coherent electromagnetic signal is such a coherent state, say $|\mu\rangle$, where $|\mu|^2$ is the average number of signal photons,

$$N_s = |\mu|^2 = E_s/\hbar\omega \tag{4.4}$$

and $\arg\mu$ is the phase of the oscillator (Glauber, 1963a). Such states correspond, as Glauber has shown, to minimum-uncertainty Gaussian wave-packets. When a coherent signal is superposed on thermal radiation (hypothesis H_1), the density operator of the mode is (Glauber, 1963b; Louisell, 1964b)

$$\rho_1 = (1 - e^{-w}) \exp[-w(a^+ - \mu^*)(a - \mu)]$$
$$= (\pi N)^{-1} \int \exp(-|\alpha - \mu|^2/N)\, |\alpha\rangle\langle\alpha|\, d^2\alpha \tag{4.5}$$

The problem of diagonalizing the operator $\rho_1 - \lambda\rho_0$ and calculating the minimum probability of error in this case has not been solved in general.

In the absence of thermal radiation, the receiver is in the vacuum state $|0\rangle$ under hypothesis H_0 and in the coherent state $|\mu\rangle$ under hypothesis H_1. The minimum probability of error has been given by equation (3.16), with now

$$q = 1 - |\langle 0|\mu\rangle|^2 = 1 - \exp(-N_s) = 1 - \exp(-E_s/\hbar\omega) \tag{4.6}$$

This probability of error is plotted in Fig. 1 for $\zeta = \frac{1}{2}$ as the line marked 'optimum'.

When, on the other hand, the average number of N of thermal photons in the mode is very large, an approximate solution illustrating the classical limit can be obtained. For simplicity we take the phase of the coherent signal as $\arg\mu = 0$, μ is real, and in the representation in which the operator

$$\xi = q(\omega/2\hbar)^{1/2} = \tfrac{1}{2}(a + a^+) \tag{4.7}$$

is diagonal, equation (3.5) becomes the integral equation

$$\int_{-\infty}^{\infty} [\langle\xi|\rho_1|\xi'\rangle - \lambda\langle\xi|\rho_0|\xi'\rangle] F_k(\xi')\,d\xi' = \eta_k F_k(\xi) \tag{4.8}$$

$$F_k(\xi) = \langle\xi|\eta_k\rangle$$

where

$$\langle\xi|\rho_0|\xi'\rangle = \left[\frac{2}{\pi(1+2N)}\right]^{1/2} \exp\left[-\frac{(\xi+\xi')^2}{2(2N+1)} - \tfrac{1}{2}(2N+1)(\xi-\xi')^2\right]$$

$$\langle\xi|\rho_1|\xi'\rangle = \exp[2\mu(\xi+\xi'-\mu)/(2N+1)]\langle\xi|\rho_0|\xi'\rangle \tag{4.9}$$

Hence, in the limit $N \to \infty$ of a large average number of thermal photons, ρ_0 and ρ_1 are nearly diagonal in the ξ-representation, and

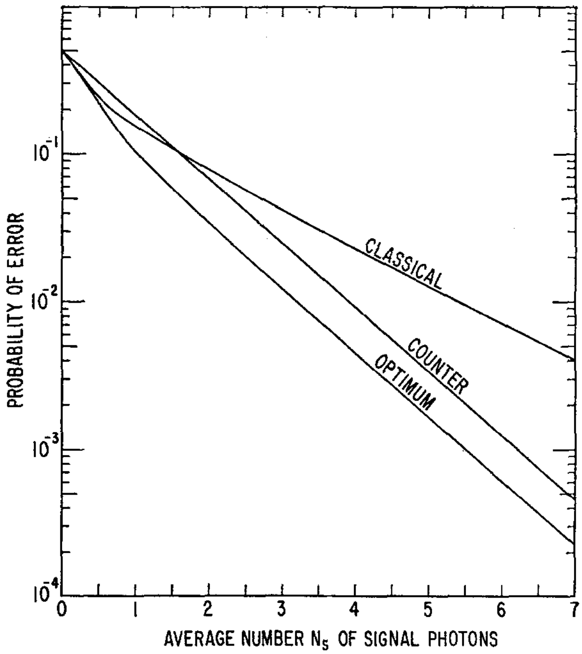

Figure 1.—The minimum average probability of detection of a coherent signal in the absence of background radiation ($\zeta = \frac{1}{2}$).

both $\langle\xi|\rho_0|\xi'\rangle$ and $\langle\xi|\rho_1|\xi'\rangle$ are proportional to $\delta(\xi - \xi')$. In the classical limit, therefore, when $\arg\mu = 0$, the best operator to measure is ξ or q, and this accords with the prescription of conventional detection theory as described by Section 2, where P_k is now equal to 0.

The observer may base his decision on the outcome of a measurement of ξ or q, choosing H_1 when a certain decision level ξ_0 is exceeded.

The probability density function of the outcome ξ of a measurement of ξ is, by equation (4.9), Gaussian under each hypothesis (Louisell, 1964b),

$$P_0(\xi) = \langle\xi|\rho_0|\xi\rangle = \left[\frac{2}{\pi(2N+1)}\right]^{1/2} \exp\left(-\frac{2\xi^2}{2N+1}\right)$$

$$P_1(\xi) = \langle\xi|\rho_1|\xi\rangle = \left[\frac{2}{\pi(2N+1)}\right]^{1/2} \exp\left[-\frac{2(\xi-\mu)^2}{2N+1}\right] \tag{4.10}$$

The decision level ξ_0 is again determined by the likelihood-ratio criterion,

$$\frac{P_1(\xi_0)}{P_0(\xi_0)} = \lambda = \frac{\zeta}{1-\zeta} \tag{4.11}$$

and the probability of error is found to be given by equation (2.8), except that the signal-to-noise ratio D is now given by (Helstrom, 1965)

$$D^2 = \frac{4\mu^2}{2N+1} = \frac{4E_s}{\hbar\omega}\tanh\left(\frac{\hbar\omega}{2kT}\right) \tag{4.12}$$

In the limit $\hbar\omega \ll kT$, this reduces to the signal-to-noise ratio in equation (2.10). This 'classical' detector has a higher probability of error than the minimum attainable.

In the limit $N = 0$, in particular, the probability of error given by equation (2.8) with equation (4.12) is greater than that given by equation (3.16) with equation (4.6). The two are compared in Fig. 1 as the lines 'optimum' and 'classical', both plotted for a prior probability $\zeta = \frac{1}{2}$.

In addition, Fig. 1 shows the probability of error of a system that merely counts the number of photons in the mode, in effect disregarding the phase information about the signal. It decides a signal is present whenever any photons at all are counted. The probability of error is then

$$P_{e,\,\mathrm{counter}} = (1-\zeta)\exp(-N_s) \tag{4.13}$$

which is plotted for $\zeta = \frac{1}{2}$ as the line marked 'counter'.

When the signal is present in many modes of the field, what corresponds to classical detection is the measurement of the operator (Helstrom, 1965)

$$\sum_k (\mu_k^* a_k + \mu_k a_k^+)/(2N_k+1)$$

where $|\mu_k|^2$ is the mean number of signal photons in the kth mode, and N_k is the mean number of thermal photons in that mode. The probability of error is again given by equation (2.8), and when the signal occupies so narrow a band of frequencies that the N_k's can be taken equal, the effective signal-to-noise ratio D is again that given by equation (4.12). All three curves in Fig. 1 are valid for such a narrowband coherent multimode signal when thermal radiation is absent ($N = 0$) and $\zeta = \frac{1}{2}$. For N not equal to zero, the optimum detection operator Π has not been determined.

5. *The Detection of Coherent Signals of Random Phase*

At optical frequencies the absolute phase of a received signal is unlikely to be known to the observer. It is, therefore, useful to determine the detectability of the signal, as expressed by the minimum average probability of error, in the least favorable situation when the phase φ of the signal is a random variable distributed uniformly over the range $0 \leqslant \varphi < 2\pi$.

How the classical receiver of Section 2 should be modified in order to detect a narrowband signal of random phase is described in books on signal detection theory, where it is shown that the minimum average probability of error attainable by such a receiver is (Helstrom, 1960b)

$$P_{\mathrm{e,\,min}} = \zeta \exp(-b^2/2) + (1-\zeta)[1 - Q(D, b)] \tag{5.1}$$

Here

$$Q(D, b) = \int_b^{\infty} x \exp[-\tfrac{1}{2}(x^2 + D^2)] I_0(Dx)\, dx \tag{5.2}$$

is Marcum's Q-function, b is a normalised decision level given by the likelihood-ratio equation

$$\lambda = \zeta/(1-\zeta) = \exp(-D^2/2) I_0(Db) \tag{5.3}$$

and $D = (2E_s/kT)^{1/2}$ is the same signal-to-noise ratio as before. We now turn to the quantum-mechanical problem.

Under hypothesis H_0, the quantum-mechanical density operator of the receiver will be given, in thermal equilibrium, by the multimode counterpart of equation (4.1),

$$\rho_0 = \int \cdots \int \exp\left(-\sum_k |\alpha_k|^2/N_k\right) |\{\alpha_k\}\rangle\langle\{\alpha_k\}| \prod_k (d^2\alpha_k/\pi N_k) \tag{5.4}$$

$$N = [\exp(\hbar\omega_k/kT) - 1]^{-1}$$

where $|\{\alpha_k\}\rangle$ is a simultaneous right-eigenket of the annihilation operators a_k of the modes, and N_k is the mean number of thermal photons in the kth mode. When a signal of overall phase φ is present, having mean complex amplitude $\mu_k(\varphi) = \mu_k e^{i\varphi}$ in the mode k, the density operator is

$$\rho_1(\varphi) = \int \cdots \int \exp\left[-\sum_k |\alpha_k - \mu_k(\varphi)|^2/N_k\right] |\{\alpha_k\}\rangle\langle\{\alpha_k\}| \prod_k (d^2\alpha_k/\pi N_k) \tag{5.5}$$

Here $|\mu_k|^2$ is the mean number of signal photons in mode k, and the phases $\arg\mu_k$ are assumed known. The density operator under hypothesis H_1 is now the average of $\rho_1(\varphi)$ over the common unknown phase φ,

$$\rho_1 = \int_0^{2\pi} \rho_1(\varphi)\, d\varphi/2\pi \tag{5.6}$$

Under the assumption that the signal occupies only a narrow band of frequencies, the numbers N_k are nearly equal for all modes in which the amplitudes $|\mu_k|$ differ significantly from zero, $N_k \equiv N$. The remaining modes can be disregarded. At any point of time, one can introduce a new set of mode operators b_k, b_k^+ by a unitary transformation (Helstrom, 1967b),

$$\begin{gathered} b_k = \sum_m V_{km} a_m \\ b_k^+ = \sum_m V_{km}^* a_m^+ = \sum_m a_m^+ (V^+)_{mk} \\ \mathbf{V} = \|V_{km}\|, \qquad \mathbf{V}\mathbf{V}^+ = \mathbf{I} \end{gathered} \tag{5.7}$$

where $\mathbf{I}$ is the identity matrix. The b_k's, b_k^+'s obey the same commutation rules as the a_k's, a_k^+'s. The right eigenkets $|\{\beta_n\}\rangle$, defined by

$$b_k|\{\beta_m\}\rangle = \beta_k|\{\beta_m\}\rangle \tag{5.8}$$

have the same properties as the coherent states $|\{\alpha_n\}\rangle$, and can be used to express the density operators ρ_0 and ρ_1 as in equations (5.4) to (5.6). In particular, because of the unitarity of the transformation $\mathbf{V}$ the density operator ρ_0 can be written as

$$\rho_0 = \int \cdots \int \exp\left(-\sum_k |\beta_k|^2/N\right) |\{\beta_m\}\rangle\langle\{\beta_m\}| \prod_m (d^2\beta_m/\pi N) \tag{5.9}$$

The unitary transformation V is so chosen that

$$V_{1m} = N_s^{-1/2}\, \mu_m^*, \qquad (V^+)_{m1} = N_s^{-1/2}\, \mu_m \tag{5.10}$$

where

$$N_s = \sum_k |\mu_k|^2 \tag{5.11}$$

is the average total number of signal photons in the receiver. The remaining rows of the matrix $\mathbf{V}$ are orthogonal to the vector $\{\mu_k\}$

$$\sum_m V_{km}\mu_m = N_s^{1/2}\delta_{1k} \tag{5.12}$$

otherwise they are arbitrary. The density operator under hypothesis H_1 is then given by equation (5.6), with now

$$\rho_1(\varphi) = \int \cdots \int \exp\left(-|b_1 - N_s^{1/2}e^{i\varphi}|^2/N - \sum_{k\neq 1} |\beta_k|^2/N\right) \times$$
$$\times\ |\{\beta_m\}\rangle\langle\{\beta_m\}| \prod_m (d^2\beta_m/\pi N) \tag{5.13}$$

We can therefore treat the decision problem as involving a single oscillator whose creation and annihilation operators are b_1^+ and b_1. The density operators ρ_0 and ρ_1 factor into an operator depending only on b_1 and b_1^+ and an operator common to ρ_0 and ρ_1. We can discard the latter factor without affecting the detectability of the signal. After we average $\rho_1(\varphi)$ over the unknown phase φ, we find

$$\rho_1 = \int \exp[-(|\beta_1|^2 + N_s)/N]\, I_0(2|\beta_1|N_s^{1/2}/N)\, |\beta_1\rangle\langle\beta_1|\, d^2\beta_1/\pi N \tag{5.14}$$

a trace having been taken with respect to the states of the remaining oscillators. As the P-representations of both ρ_0 and ρ_1 involve only $|\beta_1|$, they are simultaneously diagonal in the number representation based on the operator $n = b_1^+b_1$. The optimum detector will therefore measure the operator

$$n = b_1^+b_1 = N_s^{-1}\left(\sum_k \mu_k a_k^+\right)\left(\sum_m \mu_m^* a_m\right) \tag{5.15}$$

In the n-representation the diagonal matrix elements of the density operators are (Lachs, 1965)

$$P_{0m} = N^m/(N+1)^{m+1}$$
$$P_{1m} = \frac{1}{N+1}\left(\frac{N}{N+1}\right)^m \exp\left(-\frac{N_s}{N+1}\right) L_m\left(-\frac{N_s}{N(N+1)}\right) \tag{5.16}$$

where $L_m(x)$ is the mth Laguerre polynomial. The receiver decides that a signal is present whenever it counts a number m of photons for which

$$\frac{P_{1m}}{P_{0m}} = \exp\left(-\frac{N_s}{N+1}\right) L_m\left(-\frac{N_s}{N(N+1)}\right) \geqslant \lambda = \frac{\zeta}{1-\zeta} \tag{5.17}$$

Let n_0 be the least integer for which this is true. Then the minimum probability of error is

$$P_{\mathrm{e,min}} = \zeta \sum_{m=n_0}^{\infty} P_{0m} + (1-\zeta)\sum_{m=0}^{n_0-1} P_{1m} \tag{5.18}$$

The first term of this equation is $\zeta[N/(N+1)]^{n_0}$; for the second there is no concise expression. In the limit $N=0$ ($\zeta=\frac{1}{2}$) one obtains the minimum probability of error plotted in Fig. 1 as the line marked 'counter' and given by equation (4.13).

When the expected number of photons under hypothesis H_1 is large, the distribution P_{1m} can, by using asymptotic forms of the Laguerre functions, be written approximately as

$$P_{1m} \cong N^{-1}\exp[-(m+N_s)/N]\, I_0[2\sqrt{(N_s m)}/N] \tag{5.19}$$

and $P_{0m} \cong N^{-1}\exp(-m/N)$. The minimum probability of error is now given by equation (5.1), with the signal-to-noise ratio $D=(2N_s^2/N)^{1/2}$, which in the classical limit becomes, by the Planck formula, $D=(2E_s/kT)^{1/2}$. By these results the detectability of a narrowband signal of random phase is completely specified quantum-mechanically as well as classically.

References

Chandrasekhar, S. (1943). Stochastic Processes in Physics and Astronomy. *Review of Modern Physics*, **15**, 1–89. See Chapter II, Section 3, pp. 27–30.

Glauber, R. J. (1963a). Coherent and Incoherent States of the Radiation Field. *Physical Review*, **131**, 2766–2788. See Section VIII, pp. 2779–2781.

Glauber, R. J. (1963b). Coherent and Incoherent States of the Radiation Field. *Physical Review*, **131**, 2766–2788. See Section IX, pp. 2781–2784.

Helstrom, C. W. (1960a). *Statistical Theory of Signal Detection*, Chapter II, pp. 57–83. Pergamon Press, Ltd., Oxford.

Helstrom, C. W. (1960b). *Statistical Theory of Signal Detection*, Chapter V, pp. 129–165. Pergamon Press, Ltd., Oxford.

Helstrom, C. W. (1965). Quantum Limitations on the Detection of Coherent and Incoherent Signals. *Transactions of the IEEE*, **IT-10**, 482–490.

Helstrom, C. W. (1967a). Detection Theory and Quantum Mechanics. *Information and Control*, **10**, 254–291.

Helstrom, C. W. (1967b). Quasi-Classical Analysis of Coupled Oscillators. *Journal of Mathematics and Physics*, **8**, 37–42.

Helstrom, C. W. (1967c). Detection Theory and Quantum Mechanics, II. Submitted to *Information and Control.*

Lachs, G. (1965). Theoretical Aspects of Mixtures of Thermal and Coherent Radiation. *Physical Review*, **138**, B1012-B1016.

Louisell, W. H. (1964a). *Radiation and Noise in Quantum Electronics*, Section 6.6, pp. 228–233. McGraw-Hill Book Co., New York.

Louisell, W. H. (1964b). *Radiation and Noise in Quantum Electronics*, Section 6.11, pp. 245–248. McGraw-Hill Book Co., New York.

Middleton, D. (1960). *Statistical Communication Theory*, Chapter 19, pp. 801–833. McGraw-Hill Book Co., New York.